

Ricci-Recurrent Structures in P-Sasakian Manifolds: A Geometric Analysis of Curvature Constraints

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Abstract:

This paper investigates the geometric and curvature-related implications of Ricci-recurrent structures within the framework of P-Sasakian manifolds an important subclass of contact metric manifolds. By imposing a recurrence condition on the Ricci tensor, we examine whether such structures can be consistently realized within the intrinsic geometry of P-Sasakian spaces. Utilizing differential geometric tools, including covariant differentiation and the Ricci identity, we derive integrability conditions and curvature relations. A central geometric obstruction is demonstrated, showing that Ricci recurrence is fundamentally incompatible with the defining structure of P-Sasakian manifolds. These findings contribute to the broader classification of special Riemannian manifolds and highlight the inherent rigidity of P-Sasakian geometry under curvature constraints.

1. Introduction:

In the realm of differential geometry, the study of manifolds equipped with additional geometric structures has led to significant advancements in understanding curvature and topological behavior. Among such structures, P-Sasakian manifolds represent a rich and well-studied class that generalizes contact metric and K-contact manifolds. Defined by a Riemannian metric tensor field, a characteristic vector field, and a contact 1-form, P-Sasakian manifolds satisfy compatibility and integrability conditions that endow them with distinct geometric character[9].

In parallel, Ricci-recurrent manifolds where the Ricci tensor satisfies a specific recurrence relation involving a non-vanishing 1-form have attracted attention for their relevance to generalized symmetric spaces, semi-symmetric structures, and various models in theoretical physics. These manifolds exhibit a form of curvature memory that imposes additional constraints on geometric structures.

This paper explores the potential coexistence of these two geometric notions. Specifically, we investigate whether a P-Sasakian manifold can admit a Ricci-recurrent structure. Through a detailed analysis of curvature identities and differential geometric properties, we derive necessary conditions for Ricci recurrence and assess their compatibility with the intrinsic structure of P-Sasakian manifolds. The analysis ultimately leads to a strong non-existence result.

1.1 Research Objectives

The principal objectives of this research are as follows:

1. To define and formalize Ricci-recurrent structures within the framework of P-Sasakian geometry, using the Levi-Civita connection to articulate the recurrence condition on the Ricci tensor.
2. To analyze the geometric consequences of Ricci recurrence in this context, including the behavior of covariant derivatives and the integrability of the associated recurrence 1-form.
3. To establish curvature identities based on the Ricci identity and evaluate their implications for the structure and behavior of the Ricci tensor in P-Sasakian manifolds.

2. Ricci-Recurrent P-Sasakian Manifolds:

Ricci-recurrent manifolds constitute an important class of semi-symmetric spaces in differential geometry. When this concept is extended to P-Sasakian manifolds a class of contact metric manifolds characterized by specific curvature identities it leads to interesting geometric constraints. This section explores the implications of Ricci recurrence within the framework of P-Sasakian geometry.

Definition 2.1. Ricci-Recurrent P-Sasakian Manifold:

A P-Sasakian manifold M^n is said to be Ricci-recurrent if its Ricci tensor $R_{\alpha\beta}$ satisfies the recurrence condition[11]:

$$(2.1) \quad \nabla_\gamma R_{\alpha\beta} = b_\gamma R_{\alpha\beta},$$

where ∇_γ denotes the Levi-Civita connection, and b_γ is a non-zero 1-form known as the recurrence vector field. This condition implies that the covariant derivative of the Ricci tensor along any direction is proportional to the tensor itself, thereby generalizing the notion of Ricci-symmetric spaces. Such manifolds are of particular interest in the study of curvature structures and their evolution under geometric flows.

Contracting equation (2.1) with the inverse metric tensor $g^{\alpha\beta}$ yields the corresponding recurrence relation for the Ricci operator $R^{\alpha\beta} = g^{\alpha\tau} g^{\beta\rho} R_{\tau\rho}$:

$$(2.2) \quad \nabla_\gamma (R_{\alpha\beta} R^{\alpha\beta}) = 2b_\gamma R_{\alpha\beta} R^{\alpha\beta},$$

Differentiation of this condition, together with the torsion-free property of the Levi-Civita connection, leads to the integrability condition:

$$(2.3) \quad (R_{\alpha\beta} R^{\alpha\beta})(\nabla_\delta b_\gamma - \nabla_\gamma b_\delta) = 0,$$

This yields two significant cases:

•Case I: If $R^{\alpha\beta} = 0$, then

$$(2.4) \quad R_{\alpha\beta} = 0, \text{ which indicating that the manifold is Ricci-flat.}$$

•Case II: If $\nabla_\delta b_\gamma = \nabla_\gamma b_\delta$, then b_δ is a closed 1-form and hence locally exact. In this scenario, the recurrence vector is a gradient field, and the manifold is referred to as a gradient Ricci-recurrent P-Sasakian manifold.

2.1. Geometric Consequences of Ricci-Recurrent P-Sasakian Manifolds:

The geometric structure of P-Sasakian manifolds imposes further restrictions under Ricci recurrence. In this subsection, we examine the implications of the recurrence condition (2.1) in light of the intrinsic curvature properties of such manifolds.

Using the relations $\nabla_\delta b_\gamma = \nabla_\gamma b_\delta$ in equation (2.1), we obtain the identity:

$$(2.5) \quad \nabla_\delta \nabla_\gamma R_{\alpha\beta} - \nabla_\gamma \nabla_\delta R_{\alpha\beta} = 0,$$

Applying the Ricci identity leads to:

$$(2.6) \quad R_{\alpha\rho} R_{\delta\gamma\beta}^\rho + R_{\beta\rho} R_{\delta\gamma\alpha}^\rho = 0,$$

By transvecting equation (2.6) with $\xi^\delta \xi^\gamma$, we derive:

$$(2.7) \quad R_{\gamma\alpha} = -(n-1)g_{\gamma\alpha},$$

Substituting this result into equation (2.1) yields:

$$R_{\alpha\beta} = 0.$$

This result leads us to the following observations:

Lemma 2.1:

In a Ricci-recurrent P-Sasakian manifold, the Ricci curvature tensor vanishes identically.

Lemma 2.2:

In a P-Sasakian manifold, the Ricci curvature tensor is always non-zero.

From Lemmas-2.1 and 2.2, we deduce the following fundamental incompatibility:

Theorem 2.1:

A P-Sasakian manifold cannot admit a Ricci-recurrent structure.

Proof:

In view of the Lemma-2.2 the Ricci curvature tensor never vanishes where as for a Ricci-recurrent manifold it vanishes. Hence in view of Lemma-2.1 a P-Sasakian manifold can not be a Ricci recurrent one.

Theorem 2.2:

Let ξ^p be a non-zero vector field on a differentiable manifold. Then the Ricci tensor $R_{\alpha\beta}$ satisfies the following condition: $R_{\alpha\rho}\eta_\beta + R_{\beta\rho}\eta_\alpha = 0$.

Proof:

By transvecting equation (2.6) with $\xi^\delta\xi^\gamma$, we get

$$(2.8) \quad R_{\alpha\beta} - R_{\beta\rho}\eta_\alpha\xi^\rho + R_{\beta\alpha} - R_{\alpha\rho}\eta_\beta\xi^\rho = 0,$$

If $R_{\alpha\beta}$ is skew-symmetric and also $\xi^p \neq 0$ then we obtain

$$(2.9) \quad R_{\alpha\rho}\eta_\beta + R_{\beta\rho}\eta_\alpha = 0.$$

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